Pedagogical Reflections on Color Confinement in Chromostatics

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Abstract

Abelian and nonabelian gauge invariant states are directly compared to revisit how the unconfined abelian theory is expressed. It is argued that the Yang-Mills equations have no obvious physical content apart from their relation to underlying physical states. The main observation is that the physical states of electrostatics can be regarded as point charges connected by a uniform superposition of all possible Faraday lines. These states are gauge invariant only in the abelian case.

After the initial wonder and awe that students experience on learning that the nuclear force can be fundamentally described with a nonabelian gauge theory, there is a natural desire to solve the equations—that is the Yang-Mills equations. There is a naive expectation that we can calculate a chromostatic potential energy for an arrangement of quark color charge in the same way as for electric charge in the Maxwell equations. Ideally, we expect to find a confining potential that would explain why unbound quarks possessing the color charge are not observed. Additionally, the conditions under which deconfinement could occur should somehow emerge from the equations.

Unfortunately, it seems that the simplest solutions have no relevance to physical hadrons. General solutions to the nonlinear field equations cannot be obtained from linear combinations of known solutions. Going to the quantized theory this implies that free quanta of the gauge fields (gluons) do not exist in the same sense as photons. In trying to understand this situation, a number of plausible explanations of color confinement have been studied. For examples, see [1–4]. It has even been suggested that the confinement problem itself can be eliminated by recognizing that it is an artifact of false expectations of nonabelian gauge theories [4]. At least at low energies, Feynman diagrams with gluon exchange are conceptually misleading and physically wrong when taken individually.

Therefore, we first want to show how the naive expectation of calculating a classical potential goes wrong. Then we try to understand color confinement as well as what is meant by deconfinement by directly comparing the nonabelian and abelian cases in the

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simplest situation. A way to make the comparison direct, is to consider the abelian case as an $SO(2)$ projection of $SO(3)$. The gauge transformations in the abelian theory are then viewed as local rotations about the third axis—quantities carrying the third free index are invariant with respect to such rotations.

Consider the Yang-Mills Gauss' law for static color electric fields $\mathbf{E}^b(\mathbf{x})$ and vector potentials $\mathbf{A}^b(\mathbf{x})$ (b = 1,2 or 3):

$$\nabla \cdot \mathbf{E}^a + g\epsilon_{abc}\mathbf{A}^b \cdot \mathbf{E}^c = \rho^a \tag{1}$$

where g is the coupling constant and the antisymmetric ϵ_{abc} gives the $SU(2)$ structure constants (summation over the repeated indices is implied). The color electric energy associated with charge distribution $\rho^a(\mathbf{x})$ is

$$V = \frac{1}{2}\int d^3x\, \mathbf{E}^a \cdot \mathbf{E}^a \ . \tag{2}$$

Also, Eq. (1) is gauge covariant and V is gauge invariant with respect to the local $SO(3)$ transformations:

$$\rho'^a = R_{ab}\rho^b, \quad \mathbf{E}'^a = R_{ab}\mathbf{E}^b \tag{3}$$

and

$$\mathbf{A}'^a = R_{ab}\mathbf{A}^b - \frac{1}{2g}\epsilon_{abc}R_{bd}\nabla R_{cd} \tag{4}$$

where $R_{ab}(\mathbf{x})R_{ac}(\mathbf{x}) = \delta_{bc}$.

A standard approach of electrostatics would be to solve the differential equation (Eq. (1)) and evaluate the integral of Eq. (2). Of course this approach gives the Coulomb potential for point charges $\rho^3(\mathbf{x}) = q_1\delta^3(\mathbf{x}-\mathbf{x}_1)+q_2\delta^3(\mathbf{x}-\mathbf{x}_2)$ in the abelian case where all the indices in Eqs. (1) and (2) are replaced with a 3. For the nonabelian case we are immediately faced with a puzzle—physically, what is ρ^a? Even for the simplest situation in the classical theory there is already a puzzle. For instance, trying to define two point charges according to $\rho^a(\mathbf{x}) = q_1^a\delta^3(\mathbf{x}-\mathbf{x}_1)+q_2^a\delta^3(\mathbf{x}-\mathbf{x}_2)$ is not physically meaningful because q_1^a and q_2^a can be arbitrarily and independently rotated with a gauge transformation. One could argue that specifying q_1^a, q_2^a and $\mathbf{A}^b(\mathbf{x})$ together would be mathematically sufficient to determine V, but the physical significance of V would still be unclear without some physical gauge invariant entity (such as electric charge). It is true that ΔV is the work required to change between different spatial configurations of ρ^a and $\mathbf{A}^b$, but spatial configurations of what? It's as if we are given a geographer's data on coastline locations that are hopelessly entangled with a redefinition of latitude and longitude at each point on the map. The observable changes (in climate, vegetation, *etc.*) associated with each data set might be known, but the existence of actual coastlines is hidden. Calculating the energy associated with different data sets i.e. ρ^a and $\mathbf{A}^b$ does not solve the puzzle of what physically exists within data. In fact, it is probably misleading because if we set $\mathbf{A}^b = \mathbf{0}$ and $\rho^a(\mathbf{x}) = q_1^a\delta^3(\mathbf{x}-\mathbf{x}_1)+q_2^a\delta^3(\mathbf{x}-\mathbf{x}_2)$ we obtain the Coulomb potential as in electrostatics–yet this time with nothing that can legitimately be identified as physical sources. In a similar sense, recall that the local gravitational energy

density is not a meaningful observable because it always can be transformed to zero with local geodesic coordinates [5]. Therefore, instead of approaching chromostatics in the same spirit as electrostatics, let's turn the problem around by respecting the nonabelian gauge invariance of physical states and then show how electrostatics is recovered for the abelian theory.

To begin, let's take the point of view that the nonabelian charge density of Eq. (1) only makes sense in terms of an operator acting on some physical state. The simplest choice for our discussion is to work with charges in the adjoint representation of $SU(2)$

$$\rho^a(\mathbf{x}) = g\, a_b^\dagger(\mathbf{x}) T^a_{bc} a_c(\mathbf{x}) \tag{5}$$

where $T^a_{bc} = i\epsilon_{bac}$ and $a_b^\dagger$ and a_c are the creation and annihilation operators for point charges with respect to a gauge invariant state $|0\rangle$ such that $a_c|0\rangle = 0$. Also, the operators satisfy the commutation relations

$$[a_a(\mathbf{x}), a_b^\dagger(\mathbf{x}')] = \delta_{ab}\delta^3(\mathbf{x}-\mathbf{x}')\ . \tag{6}$$

Since the creation and annihilation operators transform the same way as the quantities in Eq. (3), their color indices must be contracted locally to form a gauge invariant operator. Geometrically, the simplest gauge invariant state contains two point charges and is created by the operator

$$\Pi[\gamma] = a_a^\dagger(\mathbf{x}(1)) U_{ab}[\gamma] a_b^\dagger(\mathbf{x}(0))\ . \tag{7}$$

Such a state represents spatially separated point charges at $\mathbf{x}(0)$ and $\mathbf{x}(1)$ connected with the Wilson line along some curve $\gamma : \mathbf{x} = \mathbf{x}_\gamma(s)$. The parameterization used for any curve γ can arbitrarily be taken on the interval $0 \le s \le 1$. The path ordered exponential $U_{ab}[\gamma]$ is defined according to

$$\left(\delta_{ab}\frac{d}{ds} - igT^c_{ab}\mathbf{A}^c\left(\mathbf{x}_\gamma(s)\right)\cdot\frac{d\mathbf{x}_\gamma(s)}{ds}\right) U_{bd}\left[\gamma_0^s\right] = 0 \tag{8}$$

where

$$U_{ab}[\gamma_0^0] = \delta_{ab} \ \text{ and } \ U_{ab}[\gamma] \equiv U_{ab}[\gamma_0^1]\ . \tag{9}$$

Notice that a similar $\Pi[\gamma]$ creates a gauge invariant two charge state in the abelian case. The corresponding abelian forms of Eqs. (5-9) are obtained by restricting all the lower color indices to 1 or 2, or equivalently by inserting the projection $(T^3T^3)_{ab}$ between each matrix product. Therefore, even for the abelian case, the physical charges of a gauge invariant state require the nonlocal presence of field lines. As thoroughly explained in a series of papers by Lavelle and McMullan (for example see [6]), this is the point of view of Dirac [7]. Dirac showed that the operator that creates a physical electron state, must create an electron together with its Coulomb field.

In order to satisfy Gauss' law of Eq. (1) with an operator valued source we require that

$$\left[\nabla\cdot\mathbf{E}^a + g\epsilon_{abc}\mathbf{A}^b\cdot\mathbf{E}^c - g a_b^\dagger(\mathbf{x})T^a_{bc}a_c(\mathbf{x}), \Pi[\gamma]\right] = 0 \tag{10}$$

which implies the following commutation relation for the path ordered exponential

$$\left[\nabla\cdot\mathbf{E}^a(\mathbf{x})+g\,\epsilon_{abc}\mathbf{A}^b(\mathbf{x})\cdot\mathbf{E}^c(\mathbf{x}),\,U_{mn}[\gamma]\right]= \\ ig\delta^3\left(\mathbf{x}-\mathbf{x}(0)\right)U_{mb}[\gamma]\epsilon_{abn}-ig\delta^3\left(\mathbf{x}-\mathbf{x}(1)\right)\epsilon_{amc}U_{cn}[\gamma]\ . \tag{11}$$

It can be shown that Eq. (11) follows from the definition of Eq. (8) if we assume that

$$[\mathbf{E}^a(\mathbf{x}),U_{mn}[\gamma]]=ig\int_0^1 ds\,\frac{d\mathbf{x}_\gamma(s)}{ds}\,\delta^3\left(\mathbf{x}-\mathbf{x}_\gamma(s)\right)U_{mb}[\gamma_s^1]iT^a_{bc}U_{cn}\left[\gamma_0^s\right]\ . \tag{12}$$

It is interesting to notice that Eq. (12) would follow directly from Eq. (8) by assuming the canonical relations $[E^a_j(\mathbf{x}),A^b_k(\mathbf{x}')]=i\delta_{jk}\delta_{ab}\delta^3(\mathbf{x}-\mathbf{x}')$. This is consistent with the fact that Eq. (10) is a statement of the gauge invariance of $\Pi[\gamma]$. The fact that the chromostatic fields $\mathbf{E}^a$ are not directly observable because they are not gauge invariant suggests that Eq. (12) is as far as we can go in terms of *solving* Eq. (1). We get a gauge invariant expression by contracting the color indices in a double commutator using Eq. (12) twice to get

$$\left[E^a_j(\mathbf{x}),\left[E^a_j(\mathbf{x}),\Pi[\gamma]\right]\right]=2\left|g\boldsymbol{\Delta}\left(\mathbf{x},\gamma\right)\right|^2\Pi[\gamma] \tag{13}$$

where the factor of two on the right is the $SO(3)$ representation dependent factor $L(L+1)$ with $L=1$. For $SO(2)$ the factor itself is simply one. Also, the vector quantity on the right can be regarded as a single Faraday line [11] of the field:

$$\boldsymbol{\Delta}\left(\mathbf{x},\gamma\right)\equiv\int_0^1 ds\,\frac{d\mathbf{x}_\gamma(s)}{ds}\,\delta^3\left(\mathbf{x}-\mathbf{x}_\gamma(s)\right)\ . \tag{14}$$

Assuming a gauge invariant state $|0\rangle$ exists such that $E^a_j|0\rangle=0$, Eq. (13) can be used to show that $\Pi[\gamma]|0\rangle$ is an eigenstate of $\mathbf{E}^a\cdot\mathbf{E}^a$ and is therefore an eigenstate of V

$$V\ \Pi[\gamma]|0\rangle=\frac{1}{2}\int d^3x\,2\left|g\boldsymbol{\Delta}\left(\mathbf{x},\gamma\right)\right|^2\Pi[\gamma]|0\rangle \tag{15}$$

where the integral is proportional to the length of the curve γ. Although it is divergent, it can be rendered finite by absorbing a divergent factor into g. We have only verified that the state representing a single flux line gives a linear confining potential. No surprise—this is the most basic result of strong coupling on the lattice [8] and of string models of hadrons [9,10]. What might be a surprise, though, is that the abelian result also gives a linear confining potential that is identical to Eq. (15), but without the factor of two in the integrand. The fact that confinement occurs for both cases is conventionally understood as the strong coupling limit result of neglecting the magnetic energy [8,10]. This is true in the nonabelian case but, notice that the key difference in the abelian case is that we can go back to Eq. (12) to use the fact that T^3 now commutes with U. Two linear combinations of U and T^3U allow us to obtain $\mathbf{E}^3$ as an observable directly

$$\left[\mathbf{E}^3(\mathbf{x}),\left(U[\gamma]\mp T^3U[\gamma]\right)_{mn}\right]=\pm g\boldsymbol{\Delta}\left(\mathbf{x},\gamma\right)\left(U[\gamma]\mp T^3U[\gamma]\right)_{mn} \tag{16}$$

where the linear combination with the $-(+)$ sign corresponds to a positive (negative) charge at $\mathbf{x}(0)$ and a negative (positive) charge at $\mathbf{x}(1)$. This is dramatically different from working

with Eq. (13) where the coherence of the field lines has been lost. Operators involving products of $\Pi[\gamma]$ corresponding to more charges give an incoherent sum of squares of Faraday lines. The abelian theory is completely different at this point; because the electric field is an observable, a state corresponding to a coherent superposition of Faraday lines can be constructed with

$$P_{\pm}[f] = a_m^\dagger(\mathbf{x}(1)) \left\{ \prod_\gamma \left(U[\gamma] \mp T^3 U[\gamma] \right)^{f[\gamma]} \right\}_{mn} a_n^\dagger(\mathbf{x}(0)) \tag{17}$$

where a formal product over curves between $\mathbf{x}(0)$ and $\mathbf{x}(1)$ is represented. Notice that even though Eq. (17) involves a nonlocal contraction of indices within the product, it is still gauge invariant if

$$\sum_\gamma f[\gamma] = 1 \ . \tag{18}$$

This would be more obvious for the analogous $U(1)$ expression in terms of complex numbers instead of commuting matrices. Consistent with Eq. (17), the commutator

$$\left[\mathbf{E}^3(\mathbf{x}), P_\pm[f]\right] = \pm g \left\{ \sum_\gamma \mathbf{\Delta}\left(\mathbf{x}, \gamma\right) f[\gamma] \right\} P_\pm[f] \tag{19}$$

involves a formal average over Faraday lines between $\mathbf{x}(0)$ and $\mathbf{x}(1)$. *It is this coherent average that allows deconfinement in the abelian case.* We could speculate here that such a state could also arise in the nonabelian theory where some mechanism provides the appropriate abelian projections. Recall that for us, inserting $(T^3T^3)_{ab}$ by hand in the nonabelian expressions gives the abelian expressions.

The average of Eq. (19) can be made explicit by evaluating a "sum over curves" path integral

$$\langle\!\langle \mathbf{\Delta}\left(\mathbf{k}, \gamma\right) \rangle\!\rangle_\Lambda = \mathcal{N} \int_\gamma \mathcal{D}\left[\mathbf{x}_\gamma(s)\right] \exp\left[-\frac{\Lambda}{2} \int_0^1 ds \left(\frac{d\mathbf{x}_\gamma(s)}{ds} \right)^2 \right] \mathbf{\Delta}\left(\mathbf{k}, \gamma\right) \tag{20}$$

where we work with the Fourier transformed Faraday line

$$\mathbf{\Delta}\left(\mathbf{k}, \gamma\right) = \int_0^1 ds \, \frac{d\mathbf{x}_\gamma(s)}{ds} \exp\left(-i\mathbf{k}\cdot\mathbf{x}_\gamma(s)\right) \tag{21}$$

and of course $\mathcal{N}$ is defined by the normalization condition

$$\mathcal{N} \int_\gamma \mathcal{D}\left[\mathbf{x}_\gamma(s)\right] \exp\left[-\frac{\Lambda}{2} \int_0^1 ds \left(\frac{d\mathbf{x}_\gamma(s)}{ds} \right)^2 \right] = 1 \ . \tag{22}$$

The average in Eq. (20) corresponds to a special case of the Eq. (19) average where a single parameter Λ regulates the scale for which the curves extend throughout space. As Λ goes to zero all curves throughout space are equally weighted; as Λ goes to infinity only the line directly between the charges contributes. It is tempting to interpret Λ in terms of a string

tension, but since the line integral in the exponent in Eq. (20) is not reparameterization invariant we will resist the temptation. The practical reason for the choice of the exponent is that it is translationally invariant and can be explicitly computed from the generating functional

$$\mathcal{Z}\left[\mathbf{x}(0), \mathbf{x}(1), \mathbf{J}(s), \Lambda\right] = \int_\gamma \mathcal{D}\left[\mathbf{x}_\gamma(s)\right] \exp\left\{-\int_0^1 ds \left[\frac{\Lambda}{2}\left(\frac{d\mathbf{x}_\gamma(s)}{ds}\right)^2 - \mathbf{J}(s)\cdot\mathbf{x}_\gamma(s)\right]\right\} \ . \tag{23}$$

Evaluating the gaussian integrals of Eq. (23) gives

$$\begin{aligned}\mathcal{Z}\left[\mathbf{x}(0), \mathbf{x}(1), \mathbf{J}(s), \Lambda\right] = \exp\left\{\frac{1}{2\Lambda}\int_0^1 ds \int_0^1 dt\, \mathbf{J}(s)\cdot\mathbf{J}(t)\left[t\,\theta(s-t)\right.\right. \\ \left.\left. +s\,\theta(t-s) - st\right] + \int_0^1 ds\, \mathbf{J}(s)\cdot\left[(1-s)\mathbf{x}(0) + s\,\mathbf{x}(1)\right] - \frac{\Lambda}{2}\left(\mathbf{x}(1)-\mathbf{x}(0)\right)^2\right\} \ .\end{aligned} \tag{24}$$

It is then straight forward using Eq. (24) to compute the average in Eq. (20) to be

$$\begin{aligned}\langle\!\langle \mathbf{\Delta}\left(\mathbf{k},\gamma\right)\rangle\!\rangle_\Lambda = \int_0^1 ds\, \left[\frac{i\mathbf{k}}{\Lambda}(s-\frac{1}{2}) + \mathbf{x}(1) - \mathbf{x}(0)\right] \\ \times \exp\left\{-i\mathbf{k}\cdot\left[(1-s)\mathbf{x}(0) + s\,\mathbf{x}(1)\right] - \frac{s(1-s)}{2\Lambda}\mathbf{k}^2\right\} \ .\end{aligned} \tag{25}$$

Also, it is reassuring to notice that

$$i\mathbf{k}\cdot\langle\!\langle \mathbf{\Delta}\left(\mathbf{k},\gamma\right)\rangle\!\rangle_\Lambda = \exp\left(-i\mathbf{k}\cdot\mathbf{x}(0)\right) - \exp\left(-i\mathbf{k}\cdot\mathbf{x}(1)\right) \ , \tag{26}$$

meaning that Gauss' law for point charges is satisfied by Eq. (25) for any Λ. In the limit as $\Lambda \to 0$, the integral of Eq. (25) is dominated by contributions near the limits of integration and becomes the Fourier transform of the familiar Coulomb field

$$\lim_{\Lambda\to 0}\langle\!\langle \mathbf{\Delta}\left(\mathbf{k},\gamma\right)\rangle\!\rangle_\Lambda = \frac{i\mathbf{k}}{k^2}\left[\exp\left(-i\mathbf{k}\cdot\mathbf{x}(1)\right) - \exp\left(-i\mathbf{k}\cdot\mathbf{x}(0)\right)\right] \ . \tag{27}$$

Therefore, we find that the physical state $P_\pm[f]|0\rangle$ of two charge electrostatics that gives

$$\mathbf{E}^3(\mathbf{x})\ P_\pm[f]|0\rangle = \frac{\pm g}{4\pi}\left\{\frac{\mathbf{x}-\mathbf{x}(0)}{|\mathbf{x}-\mathbf{x}(0)|^3} - \frac{\mathbf{x}-\mathbf{x}(1)}{|\mathbf{x}-\mathbf{x}(1)|^3}\right\} P_\pm[f]|0\rangle \tag{28}$$

is the one where the curve weight factors $f[\gamma]$ are all equal. It is remarkable that the physical electric field can be viewed as a uniform collection of all possible Faraday lines. It is clear that this state is not available to the symmetry unbroken nonabelian theory because it is built from a product of path ordered exponentials that is gauge invariant only for the abelian case. Furthermore, we expect that deconfinement requires some mechanism to provide appropriate abelian projection operators that allow the nonabelian theory to make use of such a state. For instance, consider the operator $(\phi^a T^a \phi^b T^b)_{mn}$ containing a Higgs field ϕ^a that has uniformly fallen in the arbitrary color direction along the third axis. For the symmetry broken ground state, the operator becomes $|\phi_0|^2 (T^3 T^3)_{mn}$ which is precisely the projection operator we have used to convert nonabelian expressions to abelian ones.

In summary, a way has been presented in which color confinement can be understood as a restriction imposed by nonabelian gauge invariance on the use of states representing a coherent superposition of color electric flux lines. Deconfinement is realized in the abelian theory by a state that represents charges joined by a coherent average of all possible flux lines. Therefore, a practical view of confinement in the nonabelian case is that this state is not allowed. It is conceivable that operators resulting from symmetry breaking could act as an abelian projection that makes such a state allowable and results in the deconfinement of color charge. Possibly, the *deconfinement* of weak isospin in electroweak theory could be understood in this way.

For the symmetric nonabelian case of $SU(2)$ charge without the color magnetic interaction, the only consistent static states are one-dimensional idealized flux lines that join the nonabelian charges [8,10]. It can be shown that such states are eigenstates of V as long as flux lines do not intersect. Intersections result in off diagonal interactions that allow flux lines to re-associate with a new pairing of charges as in string-flip models. It may be possible that string-flip models can be understood from the static implications of gauge invariance. To make this connection it would be necessary to consider the more realistic case of spin-$\frac{1}{2}$ quarks in the fundamental representation of $SU(3)$. Ideally, the color magnetic energy and dynamical considerations should be included in some approximation.

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